

HOLOMORPHY OVER FINITE-DIMENSIONAL
COMMUTATIVE ASSOCIATIVE $\mathbb{C}$ -ALGEBRAS:
LOCAL THEORY

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Abstract

To a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ of finite-dimensional commutative associative unital Banach $\mathbb{C}$ -algebras one can associate a sheaf $\mathcal{O}_\varphi$ on the underlying topological space $|\mathcal{A}|$ of $\mathcal{A}$ consisting of $\mathcal{B}$ -valued differentiable functions f with $\mathcal{A}$ -linear differential Df . It turns out that this class of functions exhibits a theory very similar to the classical complex analysis of one variable. In this article we give only an overview of some new results concerning the fundamentals of the corresponding local theory while at the same time also strengthen various already existing results scattered throughout the literature.

Key words: analysis over commutative algebras, monogenic functions, hypercomplex analysis

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1. Introduction. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathcal{A}$ be a finite-dimensional commutative associative unital $\mathbb{K}$ -algebra endowed with the analytic topology, and let $U \subseteq \mathcal{A}$ be open. The study of differentiable functions $f: U \rightarrow \mathcal{A}$ with $\mathcal{A}$ -linear differential Df was initiated by SCHEFFERS [1], who proved corresponding analogues of the Cauchy–Riemann equations. This along with the attempts to find generalizations to non-commutative or even non-associative settings gave birth to Hypercomplex Analysis and the study of so-called monogenic functions. Roughly, one would

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distinguish in the literature between three different approaches: *S-regular*¹ functions, *F-regular*² functions, and the more recent notion of *slice* holomorphy³. In this article, we will focus solely on the *S-regularity* approach as it appears most natural⁴ to the commutative associative setting. In the case $\mathbb{K} = \mathbb{C}$, KETCHUM [2] was the first to establish the basic analogues of Cauchy's and Morera's Integral Theorems, Cauchy's Integral Formula, $\mathcal{A}$ -analyticity, and others. This in turn was later generalized to infinite-dimensional $\mathcal{A}$ by LORCH [6] and BLUM [7].

Our desire to reopen a closer look into the finite-dimensional commutative local theory is dictated not only by the somewhat unsatisfactory, outdated, and fragmented state of the subject, but also by purely geometric considerations. While smooth manifolds over Weil algebras⁵ have been extensively studied, notably by the Kazan school of geometry (see for example [3], or [4] for more recent developments) and in the setting of Weil bundles and Weil functors (see for example [5]), the complex counterpart remains to the best of our knowledge largely unexplored. In the infinite-dimensional case, the usefulness of manifolds modelled on commutative Banach algebras was recognized by KOBAYASHI [8] in the setting of mapping spaces and later by YAGISITA [9] in the setting of *continuous* families⁶ of complex manifolds. Roughly speaking, this is because the corresponding Banach manifold becomes finite-dimensional-like over a suitable (infinite-dimensional) Banach algebra $\mathcal{A}$. In the finite-dimensional case, complex $\mathcal{A}$ -manifolds (and more generally, $\mathcal{A}$ -analytic spaces) arise naturally in several ways: for example, complex tori $\mathbb{C}^n/\Lambda$ inherit such structures *for free* via coverings $\mathcal{A} \twoheadrightarrow \mathcal{A}/\Lambda$.

While some of the applications would technically require the development of a theory of *several* (possibly non-free) algebra variables, we will focus here only on the one-dimensional case, mostly because later on we will be primarily interested in complex manifolds that are curves over some such $\mathcal{A}$ (as closest relatives to Riemann surfaces). We remark that there are infinitely many isomorphism types of such algebras in dimension 7 and above (see for example [11]). Having replaced the once canonical choice of $\mathbb{C}$ by any $\mathcal{A}$, it is clear that the local theory is categorically flavoured, though we will not be pursuing this point of view in the present article. But we point out that the concept of differentiability is inherently relative and so we build homomorphisms $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ into our definition of $\mathcal{A}$ -differentiability. While this is arguably only a cosmetic modification, it is an important one with its own subtleties and encompasses relevant examples such as anti-holomorphic functions and Banach-algebra-valued holomorphic functions. In fact, the necessity of such a setup was recognized by SHURYGIN [10] in the context

¹After G. Scheffers.

²Short for Fueter-regular after R. Fueter; these are solutions of generalized Dirac operators.

³Holomorphy restricted to suitable subspaces.

⁴For example, $\text{id}_{\mathcal{A}}$ is not Fueter-regular.

⁵Finite-dimensional commutative local (Artinian) algebras $(\mathcal{A}, \mathfrak{m})$ with residue field $\mathcal{A}/\mathfrak{m} \cong \mathbb{R}$.

⁶As opposed to the complex-analytic families in Kodaira–Spencer deformation theory.

of smooth manifolds over Weil algebras. In the same vein, $\mathcal{A}$ -differentiability involves a (non-canonical) choice of an $\mathcal{A}$ -structure on a vector space V and is thus intimately related to the representation theory of $\mathcal{A}$. This point of view too is out of the scope of the present article, which only aims to give an overview of the core facts that underpin the local theory. Moreover, the *meromorphic* theory will be treated separately due to the need to explain new singularity phenomena in this framework. A substantially expanded version of this manuscript including complete proofs will appear elsewhere. Lastly, we remark that the local theory in Sections 4–6 carries through for both $\mathbb{R}$ and $\mathbb{C}$.

2. Algebraic preliminaries and notations. Let $\text{fdCAlg}_{\mathbb{K}}$ denote the category of all finite-dimensional commutative associative unital $\mathbb{K}$ -algebras with the usual morphisms. By finite-dimensionality, every $\mathcal{A} \in \text{fdCAlg}_{\mathbb{K}}$ is Artinian and

therefore $\mathcal{A} \cong \prod_{i=1}^k \mathcal{A}_i$ for some local Artinian $\mathbb{K}$ -algebras $\mathcal{A}_i$ with nilpotent maxi-

mal ideals $\mathfrak{m}_i$ and residue fields $\kappa_i \in \{\mathbb{R}, \mathbb{C}\}$ (see [12]), or $(\mathcal{A}_i, \mathfrak{m}_i, \kappa_i)$ for short, or just $(\mathcal{A}_i, \mathfrak{m}_i)$ when the residue field is understood. Denote by $\text{pr}_i: \mathcal{A} \twoheadrightarrow \mathcal{A}_i$ the i -th canonical projection and consider a generic morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\text{fdCAlg}_{\mathbb{K}}$, where

$\mathcal{B} \cong \prod_{j=1}^{\ell} (\mathcal{B}_j, \mathfrak{n}_j, \lambda_j)$. It is not too difficult to show that there exists a uniquely

determined index mapping $\tau: \{1, \dots, \ell\} \rightarrow \{1, \dots, k\}$, $j \mapsto \tau(j)$, depending on φ , such that

$$(2.1) \quad \begin{array}{ccc} \prod_{i=1}^k \mathcal{A}_i & \xrightarrow{\varphi} & \prod_{j=1}^{\ell} \mathcal{B}_j \\ \Pi_{\tau} \downarrow & \nearrow \exists_1 \bar{\varphi} & \\ \prod_{j=1}^{\ell} \mathcal{A}_{\tau(j)} & & \end{array}$$

commutes, where $\Pi_{\tau}: (a_1, \dots, a_k) \mapsto (a_{\tau(1)}, \dots, a_{\tau(\ell)})$, i.e. $\Pi_{\tau} = (\text{pr}_{\tau(1)}, \dots, \text{pr}_{\tau(\ell)})$, and $\bar{\varphi} = \times_{j=1}^{\ell} \bar{\varphi}_j$ for some morphisms $\bar{\varphi}_j: \mathcal{A}_{\tau(j)} \rightarrow \mathcal{B}_j$, $1 \leq j \leq \ell$. We remark that τ needs neither be injective, nor surjective. This factorization helps reduce the local analytic theory to the case of a morphism between two local $\mathbb{K}$ -algebras. In the *topological* category, since φ is in particular continuous, it induces a pushforward of 1-cycles $\varphi_{\#}: Z_1(|\mathcal{A}|, \mathbb{Z}) \rightarrow Z_1(|\mathcal{B}|, \mathbb{Z})$ and thus also of homology classes $\varphi_{*}: H_1(|\mathcal{A}|, \mathbb{Z}) \rightarrow H_1(|\mathcal{B}|, \mathbb{Z})$ on the underlying topological spaces. By abuse of notation we will use the same Γ to denote both a cycle $\Gamma \in Z_1(|\mathcal{A}|, \mathbb{Z})$ and its set-theoretical support.

Suppose now that $\mathbb{K} = \mathbb{C}$. Then for all $1 \leq i \leq k$ we get $\kappa_i \cong \mathbb{C}$ and $\mathcal{A}_i = \mathbb{C} \oplus \mathfrak{m}_i$ as $\mathbb{C}$ -vector spaces. Consider furthermore the case $(\mathcal{A}, \mathfrak{m}) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n})$. Then φ is automatically itself local as a result of the element-wise nilpotency of the maximal ideals. We put $\nu(\mathcal{A}) := \max\{n \in \mathbb{N} : \mathfrak{m}^n \neq 0\}$ ⁷ and $\nu := \nu(\varphi) = \min\{\nu(\mathcal{A}), \nu(\mathcal{B})\}$.

⁷Often called *height* of the local algebra $\mathcal{A}$ in the literature.

We will denote by $\iota_{\mathcal{A}}: \mathbb{C} \hookrightarrow \mathcal{A}$ the canonical inclusion and by $\sigma_{\mathcal{A}}: \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{m} \cong \mathbb{C}$ the canonical quotient projection, or *spectral retraction*⁸. Indeed $\sigma_{\mathcal{A}} \circ \iota_{\mathcal{A}} = \text{id}_{\mathbb{C}}$, and notice that φ commutes with both ι and σ . Consequently, certain parts of the local theory are dominated by spectral arguments establishing a connection to the classical complex analysis. To this effect, for any $U \subseteq \mathcal{A}$ we put $U^{\text{sp}} := \sigma_{\mathcal{A}}(U)$ and $\Gamma^{\text{sp}} := \sigma_{\mathcal{A}, \#}(\Gamma)$ for $\Gamma \in Z_1(U, \mathbb{Z})$. Finally, it is easy to see that the group of units $\mathcal{A}^{\times}$ sits inside $\mathcal{A} = \mathbb{C} \oplus \mathfrak{m}$ as the subset $\{(z, X) : z \in \mathbb{C} \setminus \{0\}, X \in \mathfrak{m}\}$. In fact, $\mathcal{A}^{\times} \cong \mathbb{C}^{\times} \times (\mathfrak{m}, +)$ as *Lie groups* via the biregular map $(z, X) \mapsto (z, \log(1 + X/z))$.

Considering the more general case of $\mathcal{A} \in \text{fdCAlg}_{\mathbb{R}}$, one can show that $\pi_0(\mathcal{A}^{\times}) = 1$ (i.e. path-connected) if and only if $\mathcal{A}$ carries a compatible complex structure, that is, $\mathcal{A} \in \text{fdCAlg}_{\mathbb{C}}$. When this is the case, one obtains for the

homotopy classes of loops in $\mathcal{A}^{\times}$ that $[\mathbb{S}^1, \mathcal{A}^{\times}] = \prod_{i=1}^k [\mathbb{S}^1, \mathbb{C}^{\times} \times \mathfrak{m}_i] = [\mathbb{S}^1, \mathbb{C}^{\times}]^k$.

Lastly, putting $\sigma_i := \sigma_{\mathcal{A}_i} \circ \text{pr}_i: \mathcal{A} \twoheadrightarrow \mathcal{A}_i \twoheadrightarrow \mathcal{A}_i/\mathfrak{m}_i \cong \mathbb{C}$ as the i -th *spectral* projection of $\mathcal{A}$, $1 \leq i \leq k$, we define similarly as before $\Sigma_{\tau} := (\sigma_{\tau(1)}, \dots, \sigma_{\tau(\ell)}): \mathcal{A} \rightarrow \mathbb{C}^{\ell}$ (depending on φ). In other words, the morphism Σ_{τ} simply selects eigenvalues of any operator $A \in \mathcal{A}$ according to φ .

3. Analytic preliminaries and notations. For any *vector* norm $\|\cdot\|$ on $\mathcal{A}$ we will denote by $\mathbb{B}_R^{\|\cdot\|}(Z_0)$ the open $\|\cdot\|$ -ball with centre $Z_0 \in \mathcal{A}$ and radius $R \geq 0$. Every $\mathcal{A} \in \text{fdCAlg}_{\mathbb{K}}$ is a Banach $\mathbb{K}$ -algebra with respect to some (hence any⁹) *submultiplicative* norm $\|\cdot\|_{\mathcal{A}}$ (or *algebra norm* for short), for example inherited from a matrix norm via a faithful representation $\rho: \mathcal{A} \hookrightarrow M_n(\mathbb{K})$. Let ϱ denote the spectral radius function on $M_n(\mathbb{K})$ and $\varrho_{\mathcal{A}}$ its restriction to $\mathcal{A}$. Accordingly, $\mathbb{B}_{\mathcal{A}}^{\text{sp}}(Z_0, R) := \{Z \in \mathcal{A} : \varrho_{\mathcal{A}}(Z - Z_0) < R\}$ and $\overline{\mathbb{B}}_{\mathcal{A}}^{\text{sp}}(Z_0, R) := \{Z \in \mathcal{A} : \varrho_{\mathcal{A}}(Z - Z_0) \leq R\}$ will denote, respectively, the open and the closed *spectral* balls of $\mathcal{A}$. Supposing further that $\mathbb{K} = \mathbb{C}$, the basic relationship between spectral radius and algebra norms can now be summarized¹⁰ as follows:

Proposition 3.1. *Let $\|\cdot\|$ denote an arbitrary algebra norm on $M_n(\mathbb{C})$. We have:*

- (1) $\forall A \in M_n(\mathbb{C}): \|A\| \geq \varrho(A)$;
- (2) $\forall A \in \text{GL}_n(\mathbb{C}): \|A^{-1}\| \geq \varrho(A)^{-1}$;
- (3) $\forall A \in M_n(\mathbb{C}) \forall \varepsilon > 0 \exists \|\cdot\| : \varrho(A) \leq \|A\| \leq \varrho(A) + \varepsilon$;
- (4) $\forall A \in M_n(\mathbb{C}): \varrho(A) = \inf\{\|A\| : \|\cdot\| \text{ algebra norm}\}$.

4. φ -differentiability. In the following we fix a non-zero morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\text{fdCAlg}_{\mathbb{K}}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

⁸Since $\mathbb{C}$ is algebraically closed, $\mathcal{A}$ is isomorphic to an upper-triangular algebra with same-entry diagonals (see [13]).

⁹By finite-dimensionality.

¹⁰We refer the reader to [14] for more details.

Definition 4.1. Let $U \subseteq \mathcal{A}$ be open, $Z_0 \in U$, and $f: U \rightarrow \mathcal{B}$ a map.

(1) f is called (Frechet) φ -differentiable at Z_0 if there exists $B \in \mathcal{B}$ such that

$$(4.1) \quad f(Z_0 + H) = f(Z_0) + B\varphi(H) + r(H),$$

where $\|r(H)\| = o(\|\varphi(H)\|)$ as $H \rightarrow 0$. In this case $(Df)(Z_0) = f'(Z_0) := B$ is the derivative of f at Z_0 .

(2) f is called φ -holomorphic at Z_0 if there exists a neighbourhood $V \ni Z_0$ such that f is φ -differentiable everywhere on V . The space of φ -holomorphic functions on U will be denoted by $\mathcal{O}_\varphi(U)$.

(3) An $\text{id}_{\mathcal{A}}$ -holomorphic function $f: U \rightarrow \mathcal{A}$ will also be called $\mathcal{A}$ -holomorphic, and we will write $\mathcal{O}_{\mathcal{A}}(U) := \mathcal{O}_{\text{id}_{\mathcal{A}}}(U)$.

Remarks 4.2.

(1) $\mathcal{O}_\varphi$ is a sheaf of $\mathcal{A}$ -algebras with a distinguished derivation given by $f \mapsto f'$.

(2) Consider $\text{fdCAlg}_{\mathbb{K}}$. If $\sigma \in \text{Aut}_{\mathbb{K}}(\mathbb{C})$, $z \mapsto \bar{z}$, then $\mathcal{O}_\sigma$ is precisely the sheaf of anti-holomorphic functions. More generally, $\text{Aut}_{\mathbb{K}}(\mathcal{A})$ for $\mathcal{A} \in \text{fdCAlg}_{\mathbb{K}}$ is almost never trivial, e.g. $\text{Aut}_{\mathbb{K}}(\mathbb{K}[x]/(x^2)) \cong \mathbb{K}^\times$. There also exist interesting non-bijective morphisms, e.g. $\mathbb{K}[x]/(x^2) \rightarrow \mathbb{K}[y]/(y^3)$, $\bar{x} \mapsto \bar{y}^2$. Another example is $\mathcal{O}_{\iota_{\mathcal{A}}}$, which is precisely the sheaf of $\mathcal{A}$ -valued ($\mathbb{C}$ -)holomorphic functions.

(3) If $\mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$, $f \in \mathcal{O}_\varphi(U)$, $g \in \mathcal{O}_\psi(V)$, $f(U) \subseteq V$, then $g \circ f \in \mathcal{O}_{\psi \circ \varphi}(U)$.

Definition 4.3. Let $U \subseteq \mathcal{A}$ be a subset. We define the following closure operators:

$$(1) \text{ Projection closure: } \widehat{U} := \Pi_\tau^{-1}(\Pi_\tau(U)) = \prod_{i=1}^k \begin{cases} \text{pr}_i(U), & \text{if } i \in \text{Im}(\tau) \\ \mathcal{A}_i, & \text{otherwise.} \end{cases}$$

$$(2) \text{ Spectral closure}^{11}: \widetilde{U} := \Sigma_\tau^{-1}(\Sigma_\tau(U)) = \prod_{i=1}^k \begin{cases} \sigma_i(U) \times \mathfrak{m}_i, & \text{if } i \in \text{Im}(\tau) \\ \mathcal{A}_i, & \text{otherwise.} \end{cases}$$

Using Eq.(2.1), one can reduce the local theory for a general morphism φ within a few steps to the local theory over a modified category whose objects are local $\mathbb{K}$ -algebras and whose morphisms are inclusions of algebras as follows:

Lemma 4.4. Let $U \subseteq \mathcal{A}$ be open, $f: U \rightarrow \mathcal{B}$ a map, and write $f = (f_1, \dots, f_\ell)$ with $f_j: U \rightarrow \mathcal{B}_j$, $1 \leq j \leq \ell$.

(1) We have $f \in \mathcal{O}_\varphi(U)$ if and only if $\forall 1 \leq j \leq \ell: f_j \in \mathcal{O}_{\varphi_j}(U)$.

(2) If $f \in \mathcal{O}_\varphi(U)$, then for every $1 \leq j \leq \ell$ the component f_j induces a unique $\bar{f}_j \in \mathcal{O}_{\bar{\varphi}_j}(\text{pr}_{\tau(j)}(U))$ so that we have the following implication of commutative diagrams:

¹¹Note that this is defined only for $\mathbb{K} = \mathbb{C}$.

$$(4.2) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi_j} & \mathcal{B}_j \\ \text{pr}_{\tau(j)} \downarrow & \nearrow \bar{\varphi}_j & \\ \mathcal{A}_{\tau(j)} & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} U & \xrightarrow{f_j} & \mathcal{B}_j \\ \text{pr}_{\tau(j)} \downarrow & \nearrow \exists_1 \bar{f}_j & \\ \text{pr}_{\tau(j)}(U) & & \end{array}$$

- (3) Furthermore, if $f \in \mathcal{O}_\varphi(U)$, then $\bar{f} := \times_{j=1}^\ell \bar{f}_j \in \mathcal{O}_{\bar{\varphi}}(\prod_{j=1}^\ell \text{pr}_{\tau(j)}(U))$ so that we have the following implication of commutative diagrams:

$$(4.3) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ \Pi_\tau \downarrow & \nearrow \exists_1 \bar{\varphi} & \\ \prod_{j=1}^\ell \mathcal{A}_{\tau(j)} & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} U & \xrightarrow{f} & \mathcal{B} \\ \Pi_\tau \downarrow & \nearrow \exists_1 \bar{f} & \\ \Pi_\tau(U) & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \hat{U} & \xrightarrow{\exists_1 \hat{f}} & \mathcal{B} \\ \Pi_\tau \downarrow & \nearrow \bar{f} & \\ \Pi_\tau(U) & & \end{array}$$

That is to say, $f \in \mathcal{O}_\varphi(U)$ extends to a $\hat{f} \in \mathcal{O}_{\bar{\varphi}}(\hat{U})$ in a canonical way.

- (4) Let $\mathfrak{a} \triangleleft \mathcal{A}$ with $\mathfrak{a} \subseteq \text{Ker } \varphi$. Let $q_{\mathfrak{a}}: \mathcal{A} \rightarrow \mathcal{A}/\mathfrak{a}$ denote the canonical quotient projection and $\varphi_{\mathfrak{a}}: \mathcal{A}/\mathfrak{a} \rightarrow \mathcal{B}$ the induced morphism. If $f \in \mathcal{O}_\varphi(U)$, then f induces a unique $f_{\mathfrak{a}} \in \mathcal{O}_{\varphi_{\mathfrak{a}}}(q_{\mathfrak{a}}(U))$ so that we have the following implication of commutative diagrams:

$$(4.4) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ q_{\mathfrak{a}} \downarrow & \nearrow \exists_1 \varphi_{\mathfrak{a}} & \\ \mathcal{A}/\mathfrak{a} & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} U & \xrightarrow{f} & \mathcal{B} \\ q_{\mathfrak{a}} \downarrow & \nearrow \exists_1 f_{\mathfrak{a}} & \\ q_{\mathfrak{a}}(U) & & \end{array}$$

Lemma 4.4 is a generalization of a result of WATERHOUSE [15], who proved a basic version of this for products of copies of $\mathbb{R}$ and $\mathbb{C}$.

Example 4.5. Consider the so-called bicomplex numbers $\mathbb{B} := \mathbb{C}[x]/(x^2 - 1)$. Since $\mathbb{B} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{C}$ -algebras, every $\mathbb{B}$ -holomorphic function is (up to a linear change of coordinates) of the form $(f(z), g(w))$ for some $(\mathbb{C})$ -holomorphic functions f and g .

5. The generalized Cauchy–Riemann–Scheffers equations. Fix a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ and let $(a_1 = 1_{\mathcal{A}}, \dots, a_n)$ and $(b_1 = 1_{\mathcal{B}}, \dots, b_m)$ be $\mathbb{K}$ -bases for $\mathcal{A}$ and $\mathcal{B}$, respectively. The $\mathcal{A}$ -module structure induced by φ allows one to define a structure tensor $(\gamma_{jk}^i)_{1 \leq j \leq n, 1 \leq i, k \leq m}$ for φ with respect to said bases:

$$(5.1) \quad \forall 1 \leq j \leq n \forall 1 \leq k \leq m: a_j b_k = \sum_{i=1}^m \gamma_{jk}^i b_i.$$

When $\varphi = \text{id}_{\mathcal{A}}$, these are just the structure constants of $\mathcal{A}$.

Proposition 5.1 (Generalized Cauchy–Riemann–Scheffers Eq.). *Let $U \subseteq \mathcal{A}$ open, $Z_0 \in U$, and $f: U \rightarrow \mathcal{B}$ a map with $f := f^1 b_1 + \dots + f^m b_m$ for some*

$f^i: U \rightarrow \mathbb{K}$, $1 \leq i \leq m$. Then f is φ -differentiable at Z_0 if and only if f is $\mathbb{K}$ -differentiable at Z_0 and

$$(5.2) \quad \forall 2 \leq j \leq n \forall 1 \leq i \leq m: \frac{\partial f^i}{\partial z^j} = \sum_{s=1}^m \gamma_{js}^i \frac{\partial f^s}{\partial z^1}$$

in Z_0 . In this case we also have $f'(Z_0) = \frac{\partial f}{\partial z^1}(Z_0)$.

Definition 5.2 (Generalized Wirtinger Operators).

$$(5.3) \quad d_j := \frac{\partial}{\partial z^j} - \varphi(a_j) \frac{\partial}{\partial z^1}, \quad 2 \leq j \leq n.$$

Lemma 5.3. Let $U \subseteq \mathcal{A}$ open and $f: U \rightarrow \mathcal{B}$ a $\mathbb{K}$ -differentiable function. Then $f \in \mathcal{O}_\varphi(U) \Leftrightarrow \forall 2 \leq j \leq n: d_j(f) = 0$.

6. φ -integrability. For $r \in \mathbb{N} \cup \{0\}$ we will denote by $\mathcal{C}^r$ (resp. $\mathcal{C}_{pw}^r$) spaces of (piece-wise) r -times continuously differentiable maps with some given domain and codomain.

Definition 6.1. Let $U \subseteq \mathcal{A}$ be open. Then $f \in \mathcal{C}^0(U, \mathcal{B})$ is called φ -integrable if f has a φ -primitive, i.e. there exists $F \in \mathcal{O}_\varphi(U)$ such that $F' = f$.

Lemma 6.2. Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$. If $f \in \mathcal{C}^0(U, \mathcal{B})$ is such that for any rectifiable $\gamma \in \mathcal{C}^0(\mathbb{S}^1, U)$

$$\oint_\gamma f(Z) dZ = 0,$$

then f is (globally) φ -integrable.

Denote by $\Omega^1(U, \mathcal{A}) := \Omega^1(U) \otimes_{\mathbb{R}} \mathcal{A}$ the space of $\mathcal{A}$ -valued differential 1-forms on U (of sufficient regularity). Notice that $dZ \in \Omega^1(U, \mathcal{A})$ and $f(Z)dZ \in \Omega^1(U, \mathcal{B})$ for any $f \in \mathcal{C}^1(U, \mathcal{B})$ via the $\mathcal{A}$ -module structure on $\mathcal{B}$ induced by φ . Much of the subsequent theory is based on the following simple, yet crucial observation, streamlining many of the proofs and distinguishing our approach from other treatments.

Proposition 6.3. Let $U \subseteq \mathcal{A}$ open, $f \in \mathcal{C}^1(U, \mathcal{B})$, and $\omega := f(Z)dZ \in \Omega^1(U, \mathcal{B})$. If $f \in \mathcal{O}_\varphi(U)$, then ω is d -closed.

Corollary 6.4 (Generalized Cauchy–Goursat). Let $U \subseteq \mathcal{A}$ open with $\pi_0(U) = 1$ and $f \in \mathcal{C}^0(U, \mathcal{B})$. If $f \in \mathcal{O}_\varphi(U)$ and $\Gamma \in Z_1(U, \mathbb{Z})$ with $[\Gamma] = 0$ in $H_1(U, \mathbb{Z})$, then

$$(6.1) \quad \int_\Gamma f(Z) dZ = 0.$$

7. The generalized index and Cauchy’s integral formula. From this point on we will assume $\mathbb{K} = \mathbb{C}$ so that $\pi_0(\mathcal{A}^\times) = 1$.

Definition 7.1. Let $\Gamma \in Z_1(|\mathcal{A}|, \mathbb{Z})$. Then $\text{Adm}_\varphi(\Gamma) := \mathcal{A} \setminus \tilde{\Gamma}$ will be called the set of φ -admissible points for Γ . Symmetrically, Γ is called admissible for $Z_0 \in \mathcal{A}$ if $Z_0 \in \text{Adm}_\varphi(\Gamma)$. If $\varphi = \text{id}_{\mathcal{A}}$, we will also write $\text{Adm}_{\mathcal{A}}(\Gamma) := \text{Adm}_{\text{id}_{\mathcal{A}}}(\Gamma)$.

Definition 7.2 (Index). Let $\Gamma \in Z_1(|\mathcal{A}|, \mathbb{Z})$ and $Z_0 \in \text{Adm}_\varphi(\Gamma)$. Then:

$$(7.1) \quad \text{Ind}_\varphi(\Gamma, Z_0) := \frac{1}{2\pi i} \int_\Gamma \frac{dZ}{\varphi(Z - Z_0)}.$$

Moreover, we also put $\text{Ind}_\mathcal{A} := \text{Ind}_{\text{id}_\mathcal{A}}$.

Lemma 7.3. Let $\Gamma \in Z_1(|\mathcal{A}|, \mathbb{Z})$ be $\mathcal{C}_{\text{pw}}^1$ -regular and $Z_0 \in \text{Adm}_\varphi(\Gamma)$.

(1) If $\mathcal{B} \xrightarrow{\psi} \mathcal{C}$ is a further morphism, then

$$(7.2) \quad \text{Ind}_{\psi \circ \varphi}(\Gamma, Z_0) = \text{Ind}_\psi(\varphi_\# \Gamma, \varphi(Z_0)).$$

(2) If φ is local and $z_0 := \sigma_\mathcal{A}(Z_0)$, then

$$(7.3) \quad \text{Ind}_\varphi(\Gamma, Z_0) = \text{Ind}_\mathcal{A}(\Gamma, z_0) = \text{ind}(\Gamma^{\text{sp}}, z_0) \in \mathbb{Z}.$$

We remark that using a combination of Eq. (2.1) and Lemma 7.3 one can show for a general $\prod_{i=1}^k (\mathcal{A}_i, \mathfrak{m}_i) \xrightarrow{\varphi} \prod_{j=1}^\ell (\mathcal{B}_j, \mathfrak{n}_j)$ that Ind_φ is valued in the obvious lattice $\mathbb{Z}^\ell \subset \mathcal{B}$.

Proposition 7.4 (Generalized Cauchy Integral Formula). Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ and $Z_0 \in \mathcal{A}$. Furthermore, let $\Delta := \Delta_r(Z_0)$ be the open polydisc around Z_0 of radius r and let $f \in \mathcal{O}_\varphi(\Delta)$. If $\gamma \in \mathcal{C}_{\text{pw}}^1(\mathbb{S}^1, \Delta)$ is admissible for Z_0 , then

$$(7.4) \quad f(Z_0) \text{Ind}_\varphi(\gamma, Z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(Z)}{\varphi(Z - Z_0)} dZ.$$

The basic presence of an index-like quantity was recognized by RIZZA [16], [17], EDENHOFER [18], and SCHPAKIVSKIY [19], but their descriptions of it are largely incomplete. Furthermore, using Proposition 7.4, we can establish a canonical analytic continuation of f to a local cylinder¹²:

Lemma 7.5. Suppose $(\mathcal{A}, \mathfrak{m}) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n})$. Let $\Delta := \Delta_r(Z_0) \subseteq \mathcal{A}$ be a polydisk and $f \in \mathcal{O}_\varphi(\Delta)$. Then for any $\gamma \in \mathcal{C}_{\text{pw}}^1(\mathbb{S}^1, \Delta)$ the function f has a φ -holomorphic continuation to $\Delta \cup (C_\gamma \times \mathfrak{m})$ given by

$$(7.5) \quad f(Z + X) := \frac{1}{2\pi i \text{Ind}_\varphi(\gamma, Z)} \oint_\gamma \frac{f(W)}{\varphi(W - Z - X)} dW$$

for all $Z \in \Delta \cap (C_\gamma \times \mathfrak{m})$ and $X \in \mathfrak{m}$, where $C_\gamma := \{z \in \mathbb{C} \setminus \gamma^{\text{sp}} : \text{ind}(\gamma^{\text{sp}}, z) \neq 0\}$.

8. φ -analyticity.

Definition 8.1. Let $U \subseteq \mathcal{A}$ be open. A function $f: U \rightarrow \mathcal{B}$ is called φ -analytic at $Z_0 \in U$ if there exists an open neighbourhood $\mathcal{N} \ni Z_0$ such that $f|_\mathcal{N}$ is given by

$$(8.1) \quad f|_\mathcal{N}(Z) = \sum_{k=0}^{\infty} B_k \varphi(Z - Z_0)^k \in \mathcal{B}[[\varphi(Z - Z_0)]],$$

¹²Also known as *prolongation* of the map f in the literature on Weil bundles.

point-wise convergent in the $\|\cdot\|_{\mathcal{B}}$ -topology for all $Z \in \mathcal{N}$. We will denote the subset of $\mathcal{B}[[\varphi(Z)]]$ of such power series by $\mathcal{B}_{\varphi}\{Z\}$ and write $\mathcal{B}\{W\} := \mathcal{B}_{\text{id}_{\mathcal{B}}}\{W\}$.

Definition 8.2 (Analytic Radius). Let $f(Z) = \sum_{k=0}^{\infty} B_k \varphi(Z)^k \in \mathcal{B}[[\varphi(Z)]]$ and let $\|\cdot\|$ be an arbitrary vector norm¹³ on $\mathcal{B}$. We put $R := \left(\limsup_{k \rightarrow \infty} \|B_k\|^{1/k} \right)^{-1} \in [0, \infty]$.

Relying on Proposition 3.1, one can now fully characterize the convergence behaviour in $\mathcal{A}\{Z\}$, and subsequently in $\mathcal{B}_{\varphi}\{Z\}$ as well, via the following:

Proposition 8.3 (Convergence Behaviour¹⁴). Let $f(Z) = \sum_{\ell=0}^{\infty} A(\ell) Z^{\ell} \in \mathcal{A}[[Z]]$.

(1) If $R > 0$, then $f(Z)$ is locally normally convergent on $\mathbb{B}'_R(0) := \bigcup_{\|\cdot\|} \mathbb{B}^{\|\cdot\|}_R(0)$, where the union is taken over all algebra norms of $\mathcal{A}$. Moreover, if $R > 0$, then in fact $\mathbb{B}'_R(0) = \mathbb{B}^{\text{sp}}_{\mathcal{A}}(0, R)$.

(2) If $\mathcal{A} = (\mathcal{A}, \mathfrak{m})$, then $f(Z)$ is divergent everywhere on $\overline{\mathbb{B}}^{\text{sp}}_{\mathcal{A}}(0, R)^c$.

Proposition 8.4 (Generalized Cauchy Transform). Let $(\mathcal{A}, \mathfrak{m}) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n})$. Let $\mathcal{S}$ be a measurable space, μ a finite $\mathcal{B}$ -valued measure on $\mathcal{S}$, $\gamma := g \oplus G: \mathcal{S} \rightarrow \mathcal{A}$ a measurable function, where $g: \mathcal{S} \rightarrow \mathbb{C}$ and $G: \mathcal{S} \rightarrow \mathfrak{m}$, and $U \subseteq \mathcal{A}$ open such that

$$(8.2) \quad \forall Z \in U: \inf_{s \in \mathcal{S}} (|g(s) - z| - \|\varphi(G(s) - X)\|_{\mathcal{B}}) > 0.$$

Then $f: U \rightarrow \mathcal{B}$ defined by

$$(8.3) \quad f(Z) := \int_{\mathcal{S}} \frac{d\mu(s)}{\varphi(\gamma(s) - Z)}$$

is φ -analytic.

It now follows from Proposition 7.4 that every $f \in \mathcal{O}_{\varphi}(U)$ is φ -analytic. A different proof of $\mathcal{A}$ -analyticity without invoking integral kernels is given in [20]. We list some important consequences of φ -analyticity.

Lemma 8.5. Let $(\mathcal{A}, \mathfrak{m}) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n})$, $U \subseteq \mathcal{A}$ open with $\pi_0(U) = 1$, and $f \in \mathcal{O}_{\varphi}(U)$.

(1) For every $Z_0 \in U$ and $X \in \mathfrak{m}$ the following generalized derivative

$$(8.4) \quad f'_{(X)}(Z_0) := \lim_{\substack{H \rightarrow X \\ H \in \mathcal{A}^{\times}}} \frac{f(Z_0 + H) - f(Z_0)}{\varphi(H)}$$

exists and gives rise to a function $\check{f}: \mathfrak{m} \times U \rightarrow \mathcal{B}$, $(X, Z) \mapsto f'_{(X)}(Z)$, polynomial in X and φ -analytic in Z for every X .

¹³Notice that in the finite-dimensional setting R is independent of the choice of a vector norm.

¹⁴Part of the convergence properties of $f \in \mathcal{A}[[Z]]$ have been already established in [2]. We remark that we also give a different proof in the extended version of the manuscript.

(2) The function $g: U \times U \rightarrow \mathcal{B}$,

$$(8.5) \quad g(Z, W) := \begin{cases} \frac{f(W) - f(Z)}{\varphi(W - Z)}, & \text{if } W - Z \in \mathcal{A}^\times \cap U \\ f'_{(W-Z)}(Z), & \text{if } W - Z \in \mathfrak{m} \cap U \end{cases}$$

is φ -holomorphic in both Z and W .

Proposition 8.6 (Generalized Morera). *Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$ and let $f \in \mathcal{C}^0(U, \mathcal{B})$. If for all simple rectifiable loops γ in U we have $\oint_\gamma f(Z) dZ = 0$, then $f \in \mathcal{O}_\varphi(U)$.*

9. The spectral homology.

Definition 9.1 (“Spectral” 1-Homology). *Let $U \subseteq \mathcal{A}$ be open. We set:*

$$(9.1) \quad H_1^{\text{sp}}(U, \mathbb{Z}) := Z_1(U, \mathbb{Z}) / \bigcap_{i \in \text{Im} \tau} (\sigma_i)_\#^{-1}(B_1(\sigma_i(U), \mathbb{Z})).$$

Definition 9.1 is the topological counterpart of Ind_φ . Furthermore, one can use Lemma 8.5 to prove the following strong form of Proposition 7.4.

Theorem 9.2 (Generalized Homological Cauchy Integral Formula). *Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$, $f \in \mathcal{O}_\varphi(U)$, and $\Gamma \in Z_1(U, \mathbb{Z})$. If $[\Gamma] = 0$ in $H_1^{\text{sp}}(U, \mathbb{Z})$, then*

$$(9.2) \quad \forall Z \in U \cap \text{Adm}(\Gamma): f(Z) \text{Ind}_\varphi(\Gamma, Z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(W)}{\varphi(W - Z)} dW.$$

A similar result was proved by Rizza [16], [17], but under much stronger topological assumptions on Γ even in the local case $\mathcal{A} = (\mathcal{A}, \mathfrak{m})$.

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